RBT Properties

- Every ~~node~~ path from root to leaf has the same number of black nodes
- No two red nodes are adjacent.

Additional common restrictions

- Root is black
- All external nodes are black

$\boxed{\text{Theorem / Claim:}}$

A red-black tree has a maximum height of $2\log(n+1)$.

Intuition:

We want to show there is some feature of the design that is allowing trees to grow <u>wide</u> while keeping them <u>shallow</u>.

The only height-related property is the black nodes being counted, and the fact that red nodes can only "interrupt" a chain of black nodes evenly through the tree.

LEMMA 1:

An RBT with root x has $n \geq 2^{bh(x)} - 1$ nodes, where $bh(x)$ is the black height of node x.

Proof:

By induction on the black height of x.

BASE CASE:

$bh(x) = 0 \iff x$ is a leaf.

$2^0 - 1 = 1 - 1 = 0$

$n = 0$

$0 \leq 0$ ✓

Statement holds.

INDUCTIVE:

Assume $n \geq 2^{bh(x)} - 1$ holds for all ~~possible nodes subtrees of x, that is, of all~~ nodes with a black height of k.

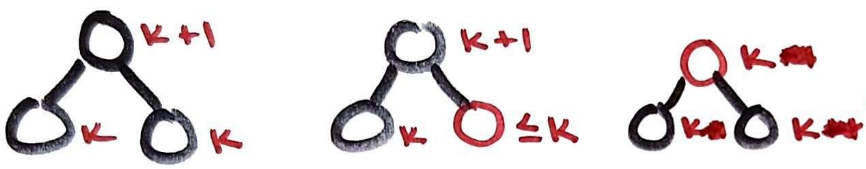
We show this holds for any node with $bh(x) = k+1$.

Let x be some node with left child L and right child R.* Then let n_R be the number of internal nodes for a subtree with root R, and similar for L.

~~Therefore~~

* We should make clear that this has implications for bh(L) and bh(R).

If $bh(x) = k+1$, then $bh(L) \leq k$ and $bh(R) \leq k$.



⇒ not actually ~~simplify~~ an option if considering k+1

Therefore:

$$
\begin{aligned}
n &= n_R + n_L + 1 \\
&\geq 2^{bh(L)} - 1 + 2^{bh(R)} - 1 + 1 \\
&= 2^{bh(L)} + 2^{bh(R)} - 1 \\
&= 2^k + 2^k - 1 \\
&= 2(2^k) - 1 \\
&= 2^{k+1} - 1 \\
&= 2^{bh(x)} - 1
\end{aligned}
$$

■

emma 2:

Any node x with height h(x) has

$$bh(x) \geq \frac{h(x)}{2}.$$

Proof:

Since red nodes cannot be consecutive, (and leaves are black, as is the root) there must be at least one black node for every red node, along a path.

Thus $bh(x) \geq h(x)/2$.

FINALLY!

THEOREM:

A red-black tree has a maximum height of $2\log(n+1)$.

Proof:

Let h be the height of an RBT with n nodes. Lemma 2 states $bh(x) = h/2$. Lemma 1 states $n \geq 2^{bh(x)} - 1$.

Therefore:

$$n \geq 2^{bh(x)} - 1$$

$$\geq 2^{h/2} - 1$$

$$\Rightarrow n+1 \geq 2^{h/2}$$

$$\Rightarrow \log(n+1) \geq h/2$$

$$\Rightarrow h \leq 2\log(n+1)$$